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Supersymmetry on the noncommutative lattice

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Abstract: Built upon the proposal of Kaplan et al. hep-lat/0206109, we construct noncommutative lattice gauge theory with manifest supersymmetry. We show that such theory is naturally implementable via orbifold conditions generalizing those used by Kaplan et al. We present the prescription in detail and illustrate it for noncommutative gauge theories latticized partially in two dimensions. We point out a deformation freedom in the defining theory by a complex-parameter, reminiscent of discrete torsion in string theory. We show that, in the continuum limit, the supersymmetry is enhanced only at a particular value of the deformation parameter, determined solely by the size of the noncommutativity.

Keywords: Lattice Quantum Field Theory, Lattice Gauge Field Theories, M(atrix) Theories, Non-Commutative Geometry.

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Contents

1. Introduction

Matrix models have played an important role in string and gauge theories. They provide a powerful constructive approach to two-dimensional quantum gravity coupled to various matters, which can also be viewed as noncritical string theories. This idea has developed into the recent conjectures that a certain type of matrix models, called as reduced models, serve as nonperturbative definitions of superstring/M theories [1, 2]. In particular there are some evidences that four-dimensional space-time is *dynamically* generated in the matrix model for type-IIB superstring theory [3].

Historically, the reduced model was introduced as an equivalent description of large-N gauge theories [4, 5, 6] in the planar limit. It can be formally obtained by dimensional reduction of $\mathrm{SU}(\infty)$ gauge theories. Recently reduced models have been reinterpreted as gauge theories (with gauge group of finite rank) on noncommutative geometry [7]–[11]. In this interpretation, the space-time coordinates and the color indices are treated on equal footing as matrix indices. The gauge invariance results from the $\mathrm{SU}(\infty)$ symmetry of the reduced model. Making the size of the matrices finite corresponds to discretizing the space-time into a lattice [9, 10, 11]. This approach was crucial for rendering field theories on noncommutative geometry accessible by Monte Carlo simulations [12]. By further imposing appropriate orbifold conditions, one can obtain a finite noncommutative torus with an arbitrary noncommutativity parameter [9]. As a particular case, one can also obtain commutative space-time. Thus, any lattice field theory (including Wilson's lattice gauge theory) can be embedded in a matrix model with a certain orbifold condition.

Recently, reduced models were shown to be also useful for constructing lattice theories with manifest supersymmetry [13]. The construction consists of two steps. First one considers the mother theory, which is a reduced model obtained from dimensional reduction of SU(N) super Yang-Mills theory *in the continuum*. (In fact the case with maximal supersymmetry corresponds to the matrix models for superstring/M theories mentioned above.) Then one imposes the orbifold condition on the mother theory and arrives at a daughter theory, which inherits part of the supersymmetry from the mother theory. Here, the orbifold condition plays a crucial role in introducing a lattice structure to the daughter theory. The idea of respecting some symmetry on the lattice and avoiding fine-tuning thereof in obtaining a supersymmetric continuum limit was also discussed in refs. [14].

In this paper, we consider generalization of Kaplan et al.'s new approach to lattice supersymmetry and show that noncommutative super Yang-Mills theories can be naturally constructed by adopting a generalized orbifold condition (similar to the consideration of ref. [9] in a different context). As an illustration, we present an explicit construction of a noncommutative U(k) gauge theory with maximal supersymmetry,[1] which is of particular importance due to its relationship to superstring/M theories [7, 8]. We also mention a close relationship to discrete torsion studied in the context of string theory on orbifold.

2. Emergent space-time out of matrix orbifolds

We begin with a brief recapitulation concerning emergent space-time out of matrices via certain orbifold projection conditions. As a toy model, consider a matrix model of $(N \times N)$ complex matrices Φ_i $(i = 1, \ldots, M)$, whose action is given by

$$S = \frac{1}{g^2}\,\mathrm{Tr}(\Phi_1 \cdots \Phi_M)\,. \tag{2.1}$$

On the matrices Φ_i, we impose 'orbifold conditions' of the following sort:

$$\Phi_i = {\omega_L}^{r_{i,a}}\,\Omega_a^\dagger \Phi_i\,\Omega_a\,. \tag{2.2}$$

Here, ω_L is a phase-factor

$$\omega_L := \mathrm{e}^{\frac{2\pi i}{L}} \qquad \text{obeying} \qquad {\omega_L}^L = 1\,,$$

and Ω_a $(a = 1, \ldots, d)$ are $N \times N$ unitary matrices. Details of the emergent space-time, including (non)commutativity, turn out to depend on specific choices of these matrices. For now, we choose them to be

$$\Omega_1 = U_L \otimes \mathbb{1}_L \otimes \cdots \otimes \mathbb{1}_L \otimes \mathbb{1}_k$$

[1] It was speculated that this sort of theories accommodates gravity as well, though they are defined on a flat (noncommutative) space-time [15].

$$
\begin{aligned}
\Omega_2 &= \mathbb{1}_L \otimes U_L \otimes \cdots \otimes \mathbb{1}_L \otimes \mathbb{1}_k \\
&\ \vdots \\
\Omega_d &= \underbrace{\mathbb{1}_L \otimes \mathbb{1}_L \otimes \cdots \otimes U_L}_{d} \otimes \mathbb{1}_k \,.
\end{aligned}
\tag{2.3}
$$

We assume that N, the matrix size, can be factorized as

$$
N = k \cdot L^d \tag{2.4}
$$

for some integer k. In the definition eq. (2.3), $\mathbb{1}_L$ and $\mathbb{1}_k$ stand for a unit matrix of the specified sizes, and U_L stands for the 'clock matrix' of size $p = L$:

$$
U_p = \begin{pmatrix} 1 & & & & \\ & \omega_p & & & \\ & & {\omega_p}^2 & & \\ & & & \ddots & \\ & & & & {\omega_p}^{p-1} \end{pmatrix} . \tag{2.5}
$$

For later convenience, we also introduce the 'shift matrix' by

$$
V_p = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix} . \tag{2.6}
$$

The pair of matrices eqs. (2.5,2.6) satisfies the well-known 't Hooft-Weyl algebra:

$$
U_p V_p = \omega_p^{-1} \, V_p U_p \qquad \text{where} \qquad \omega_p := e^{\frac{2\pi i}{p}} \,. \tag{2.7}
$$

For Φ_i's obeying the orbifold condition eq. (2.2), the set of "charge vector" $(\mathbf{r}_i)_a := r_{i,a}$, assigned uniquely for each matrix Φ_i, ought to satisfy the condition:

$$
\sum_{i=1}^{M} \mathbf{r}_i = 0 \,, \tag{2.8}
$$

else the action eq. (2.1) would vanish trivially. We refer eq. (2.8) as 'charge neutrality condition', and, in what follows, we will assume that it is always satisfied. In constructing super-symmetric gauge theories, a convenient choice of $\mathbf{r}_i$ would be to take a suitable linear combination and re-scaling of the charges associated with the R-symmetry [13].

To solve eq. (2.2), we find it convenient to introduce $(N \times N)$ unitary matrices D_a:

$$
\begin{aligned}
D_1 &= V_L^\dagger \otimes \mathbb{1}_L \otimes \cdots \otimes \mathbb{1}_L \otimes \mathbb{1}_k \\
D_2 &= \mathbb{1}_L \otimes V_L^\dagger \otimes \cdots \otimes \mathbb{1}_L \otimes \mathbb{1}_k \\
&\ \vdots \\
D_d &= \underbrace{\mathbb{1}_L \otimes \mathbb{1}_L \otimes \cdots \otimes V_L^\dagger}_{d} \otimes \mathbb{1}_k \,,
\end{aligned}
$$

which obey a different set of orbifold conditions

$$D_b = {\omega_L}^{\delta_{a,b}} \, \Omega_a^\dagger D_b \Omega_a \qquad (a, b = 1, \ldots, d)\,.$$

Then, by virtue of the 't Hooft-Weyl algebra eq. (2.7), we find a particular solution to the orbifold condition eq. (2.2) as

$$\Phi_i^{(0)} = \prod_{a=1}^{d} {D_a}^{r_{i,a}}\,.$$

We then decompose the matrices Φ_i's around the particular solution:

$$\Phi_i = \widetilde{\Phi}_i \Phi_i^{(0)}\,, \tag{2.9}$$

and find that the shifted matrices $\widetilde{\Phi}_i$'s obey 'homogeneous' orbifold condition:

$$\widetilde{\Phi}_i = \Omega_a^\dagger \widetilde{\Phi}_i \Omega_a\,. \tag{2.10}$$

As the conditions eq. (2.10) are linear equations, all we need is to construct a complete set of basis of the (finite-dimensional) solution space. For this purpose, we shall introduce $(L^d \times L^d)$ unitary matrices,[2] denoted as Z_a's:

$$\begin{aligned} Z_1 &= U_L \otimes \mathbb{1}_L \otimes \cdots \otimes \mathbb{1}_L \\ Z_2 &= \mathbb{1}_L \otimes U_L \otimes \cdots \otimes \mathbb{1}_L \\ &\vdots \\ Z_d &= \underbrace{\mathbb{1}_L \otimes \mathbb{1}_L \otimes \cdots \otimes U_L}_{d}\,. \end{aligned}$$

As defined so, $(Z_a)^L = 1$ for all $a = 1, \ldots, d$.

Evidently, any matrices of the form $Z_a \otimes M$, where M is an arbitrary $(k \times k)$ matrix, are solutions to the homogeneous orbifold condition eq. (2.10). For the special case of $k = 1$, the complete basis of the solution space is spanned by

$$J(\mathbf{p}) = \prod_{a=1}^{d} (Z_a)^{p_a} \qquad \text{where} \qquad \mathbf{p} := (p_1, \ldots, p_d)\,,$$

and p_a's take values in $0, 1, \ldots, (L-1)$. A dual basis, which turns out more convenient, is obtainable via the Fourier transformation as

$$\Delta(\mathbf{n}) := \sum_{\mathbf{p}} J(\mathbf{p})\, {\omega_L}^{\mathbf{p}\cdot\mathbf{n}} \qquad \text{where} \qquad \mathbf{n} := (n_1, \ldots, n_d)\,.$$

Here again, n_a's take values in $0, 1, \ldots, (L-1)$. For arbitrary $k > 1$, a general solution to the homogeneous orbifold condition eq. (2.10) is always expressible as

$$\widetilde{\Phi}_i = \sum_{\mathbf{n}} \Delta(\mathbf{n}) \otimes \varphi_i(\mathbf{n})\,, \tag{2.11}$$

[2]Note that, in the present case, $Z_a \otimes \mathbb{1}_k$'s turn out identical to Ω_a's, but they are distinct matrices in general cases. We will encounter such a case later when constructing noncommutative space-time.

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where $\varphi_i(\mathbf{n})$'s denote $(k \times k)$ matrix-valued functions of the d-dimensional lattice vector $\mathbf{n}$. Substituting this and eq. (2.9) into the mother theory action eq. (2.1) and utilizing the identity

$$D_a \left[\Delta(\mathbf{n}) \otimes \varphi_i(\mathbf{n})\right] D_a^\dagger = \Delta(\mathbf{n} - \hat{a}) \otimes \varphi_i(\mathbf{n}) \,,$$

we finally obtain an action of the daughter theory:

$$S = \sum_{\mathbf{n}} \mathrm{tr}\Big[\varphi_1(\mathbf{n})\varphi_2(\mathbf{n}+\mathbf{r}_1)\varphi_3(\mathbf{n}+\mathbf{r}_1+\mathbf{r}_2)\cdots\varphi_M(\mathbf{n}+\mathbf{r}_1+\cdots+\mathbf{r}_{M-1})\Big] \,. \tag{2.12}$$

Here, the symbol 'tr' is to denote the trace over the $(k \times k)$ matrices, as contrasted to the symbol 'Tr' in eq. (2.1) denoting the trace over the $(N \times N)$ matrices. Accordingly, the matrix degrees of freedom are reduced by the ratio, $N/k = L^d$, which is precisely the volume of the emergent space-time. Hence, the zero-dimensional matrix model eq. (2.1), along with the orbifold condition eq. (2.2) is the same as a (matrix) field theory on the d-dimensional lattice, where the matrix-valued functions $\varphi_i(\mathbf{n})$ brought up in eq. (2.11) are interpreted as (matrix) fields on the emergent lattice.

At the outset, we have assumed that the charge vectors obey the neutrality condition eq. (2.8). What is the reason behind the condition? What would happen if the condition is not met? Notice that, prior to imposing the orbifold condition eq. (2.2), the mother theory eq. (2.1) is actually invariant under $\mathrm{U}(N)$ rotation:

$$\Phi_i \quad \longmapsto \quad G\,\Phi_i\,G^\dagger \qquad \text{where} \qquad G \in \mathrm{U}(N) \,. \tag{2.13}$$

This invariance is compatible with the orbifold condition eq. (2.2) for Φ_i if and only if the rotation matrix G satisfies the homogeneous orbifold condition eq. (2.10) as well. In that case, in complete analogy with eq. (2.11), we can write the orbifold projected rotation matrix G as

$$G = \sum_{\mathbf{n}} \Delta(\mathbf{n}) \otimes g(\mathbf{n}) \,.$$

Inserting this back to eq. (2.13), we readily find that the $\mathrm{U}(N)$ rotation is reduced to a $\mathrm{U}(k)$ *gauge* transformation

$$\varphi_i(\mathbf{n}) \quad \longmapsto \quad g(\mathbf{n})\,\varphi_i(\mathbf{n})\,g^\dagger(\mathbf{n}+\mathbf{r}_i) \,. \tag{2.14}$$

Now, eq. (2.14) indicates that, if the lattice field $\varphi_i(\mathbf{n})$ carries no charge ($\mathbf{r}_i = 0$), it may be considered naturally as a variable residing on the lattice site $\mathbf{n}$, while those with charge ($\mathbf{r}_i \neq 0$) may be considered as a field residing on the lattice link connecting site $\mathbf{n}$ and site $(\mathbf{n} + \mathbf{r}_i)$. The latter fields are precisely the counterpart of link variables in the standard lattice gauge theory. We then learn that the daughter theory eq. (2.12) is invariant under the emergent gauge transformation eq. (2.14) if and only if the lattice links close up, viz.

$$\mathbf{n} = \mathbf{n} + \mathbf{r}_1 + \cdots + \mathbf{r}_{M-1} \,.$$

We thus learn that the charge neutrality condition eq. (2.8) is precisely what ensures the emergent daughter theory to possess the local gauge invariance eq. (2.14).

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In the above construction, the emergent space-time was commutative because all the Z_a's and hence all the $\Delta(\mathbf{n})$'s commute one another. This originates from the specific choice of the Ω_a matrices as in eq. (2.3)[3] If we adopt different choices of the Ω_a matrices, we may be able to render the emergent space-time noncommutative. What specific choices are required for the (non)commutative space-time is then an interesting question, and we obtain the answer in detail in the next sections.

3. Super Yang-Mills theory on the commutative tori

In this section, we will present a prescription of constructing theories with manifest supersymmetry on a commutative tori. We will do so in a set-up generalizable to noncommutative case. For concreteness, we will consider a $(2+1)$-dimensional super Yang-Mills theory with sixteen supercharges and gauge group $\mathrm{U}(N)$. We will first describe the commutative case in some detail in this section, and generalize it to the noncommutative case in the next section.

3.1 The mother theory

For technical convenience, we prescribe the mother theory as follows. Consider the $(3+1)$-dimensional $\mathcal{N}=4$ $\mathrm{U}(N)$ super Yang-Mills theory, which may be expressed in terms of the manifest $\mathcal{N}=1$ superfield notation (See, e.g., ref. [16] for conventions and notations). Dimensionally reducing it down to $(0+1)$ dimension, we arrive at the mother theory with the action $S=\int \mathrm{d}t\, L$, where the lagrangian L is given as

$$\begin{aligned} L &= L_g + L_\Phi + L_W, \\ L_g &= \frac{1}{16g^2}\,\mathrm{Tr}\,\Big[W^\alpha W_\alpha\Big]\Big|_{\theta\theta} + \text{h.c.}, \\ L_\Phi &= \frac{1}{g^2}\sum_{a=1}^{3}\mathrm{Tr}\,\Big[\overline{\Phi}_a e^{\mathcal{V}}\Phi_a e^{-\mathcal{V}}\Big]\Big|_{\theta\theta\bar\theta\bar\theta}, \\ L_W &= \frac{\sqrt{2}}{g^2}\,\mathrm{Tr}\,\Big[\Phi_1\Phi_2\Phi_3 - \Phi_1\Phi_3\Phi_2\Big]\Big|_{\theta\theta} + \text{h.c.}. \end{aligned} \tag{3.1}$$

$\mathcal{V}$ is a vector superfield and Φ_j are chiral superfields. W^α is a superfield containing the gauge field strength, which is made from $\mathcal{V}$. (Before dimensional reduction, L_g represents (3+1)-dimensional $\mathcal{N}=1$ pure super Yang-Mills theory, while L_Φ contains kinetic terms and gauge interaction terms of six scalar fields. The (commutator)^2-interactions among the six scalar fields come out of L_W after integrating out auxiliary fields. See appendix A.)

The mother theory eq. (3.1) inherits the SU(4) R-symmetry of the (3+1)-dimensional $\mathcal{N}=4$ super Yang-Mills theory. Under the R-symmetry group rotation, six components (real and imaginary parts) of Φ_1,Φ_2,Φ_3 transform as the antisymmetric representation $\mathbf{6}$, whereas the fermions transform chirally in the fundamental representation $\mathbf{4}$. Combined with the three extra scalar multiplets arising from dimensional reduction from $(3+1)$- to

[3] Our choice of Ω_a's in eq. (2.3) is the same as the one taken by Kaplan et al. [13].

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$(0+1)$-dimensions, the R-symmetry group of the mother theory is promoted to $G_R^{\rm mother} = {\rm Spin}(9)$. We shall be interested in the quotient of the R-symmetry group after imposing a given choice of orbifold conditions.

3.2 The daughter theory

As we are going to construct two spatial dimensions out of matrices, we shall be considering the $d = 2$ case of the orbifold condition eq. (2.2). Therefore, the size N of each matrix in the mother theory is taken as $N = k \cdot L^2$. The orbifold group in eq. (2.2) is $G_O = \mathbf{Z}_L \otimes \mathbf{Z}_L$. R-charge vectors of the mother theory fields are denoted as $\mathbf{r}_s (s = v, 1, 2, 3)$, and are assigned with integer-valued components as in the following table. This amounts to picking up three Cartan subgroup ${\rm SO}(2) \times {\rm SO}(2) \times {\rm SO}(2)$ of $G_R^{\rm mother} = {\rm Spin}(9)$, and use two independent combinations of them for the phase-rotation in the orbifold conditions eq. (2.2). As in the previous section, the orbifold group G_O lets a two-dimensional lattice emerge in the end. Hereafter, we adopt a unified notation $\Phi_v \equiv \mathcal{V}$.

Φ_s	$\mathbf{r}_s = (r_1, r_2)_s$
$\mathcal{V}$	$(0\ ,\ 0)$
Φ_1	$(+2, 0)$
Φ_2	$(-1, +1)$
Φ_3	$(-1, -1)$

Table 1: R-charge.

Imposing the orbifold conditions eq. (2.2), we obtain a daughter theory given by

$$\begin{aligned}
L_g &= \frac{1}{16g^2} \sum_{\mathbf{n}} \mathrm{tr} \left[W^\alpha(\mathbf{n}) W_\alpha(\mathbf{n}) \right]\Big|_{\theta\theta} + \text{h.c.}, \\
L_\Phi &= \frac{1}{g^2} \sum_{\mathbf{n}} \mathrm{tr} \Big[\overline{\Phi}_1(\mathbf{n}) e^{\mathcal{V}(\mathbf{n})} \Phi_1(\mathbf{n}) e^{-\mathcal{V}(\mathbf{n}+2\hat{x})} + \\
&\qquad + \overline{\Phi}_2(\mathbf{n}) e^{\mathcal{V}(\mathbf{n})} \Phi_2(\mathbf{n}) e^{-\mathcal{V}(\mathbf{n}-\hat{x}+\hat{y})} + \\
&\qquad + \overline{\Phi}_3(\mathbf{n}) e^{\mathcal{V}(\mathbf{n})} \Phi_3(\mathbf{n}) e^{-\mathcal{V}(\mathbf{n}-\hat{x}-\hat{y})} \Big]\Big|_{\theta\theta\bar{\theta}\bar{\theta}}, \\
L_W &= \frac{\sqrt{2}}{g^2} \sum_{\mathbf{n}} \mathrm{tr} \Big[\Phi_1(\mathbf{n}) \Phi_2(\mathbf{n}+2\hat{x}) \Phi_3(\mathbf{n}+\hat{x}+\hat{y}) - \\
&\qquad - \Phi_1(\mathbf{n}) \Phi_3(\mathbf{n}+2\hat{x}) \Phi_2(\mathbf{n}+\hat{x}-\hat{y}) \Big]\Big|_{\theta\theta} + \text{h.c.}
\end{aligned} \tag{3.2}$$

Here, $\hat{x}$ and $\hat{y}$ denote unit vectors along the emergent x- and y- directions. After the orbifold projection, the number of super-symmetry is reduced to four from sixteen the mother theory eq. (3.1) originally retained.

A parameter defining the daughter theory is L. If L is even, the theory comprises of two decoupled daughter theories defined on even and odd lattice sites, respectively, and thus we end up with two copies of the same theory. Here, we assume that L is odd, in which case the resulting theory is defined on the entire lattice. Using the periodicity$(n_x \sim n_x + L)$, we can interpret that n_x takes the even values

$$n_x = 0, 2, \ldots, L-1, L+1, L+3, \ldots, 2L-2 \tag{3.3}$$

for sites of n_y even, and that n_x takes the odd values

$$n_x = 1, 3, \ldots, L-2, L, L+2, \ldots, 2L-1 \tag{3.4}$$

for sites of n_y odd. As a consequence of the interpretation, all the interactions on the lattice in eq. (3.2) connect only nearest neighbor lattice sites.

The boundary conditions of the lattice fields turn out somewhat nontrivial:

$$\Phi_s(\mathbf{n}+2L\hat{x}) = \Phi_s(\mathbf{n}+L\hat{x}+L\hat{y}) = \Phi_s(\mathbf{n}). \tag{3.5}$$

In the y-direction, the simple periodic boundary condition $\Phi_s(\mathbf{n}+L\hat{y}) = \Phi_s(\mathbf{n})$ does not hold. Note that $\mathbf{n}+L\hat{y}$ does not belong to the lattice sites if $\mathbf{n}$ is a lattice site. The boundary condition eq. (3.5) indicates that the two-dimensional lattice space is a torus obtained by identifying opposite edges of a parallelogram connecting the sites $(0,0)$, $(2L,0)$, (L,L), $(3L,L)$.

To render the parallelogram isotropic, we introduce re-scaled lattice spacings for x- and y-directions as $\epsilon_x = \frac{1}{2}\epsilon$ and $\epsilon_y = \frac{\sqrt{3}}{2}\epsilon$. The two-dimensional lattice now consists of equilateral triangles. Denote site coordinates as $\mathbf{x} = (x,y) = (n_x\epsilon_x, n_y\epsilon_y)$. Denote also $\ell \equiv L\epsilon$. We also denote the lattice fields as

$$\Phi_s(\mathbf{n}) = \Phi_s(n_x,n_y) \equiv \Phi_s(x,y) = \Phi_s(\mathbf{x})\,. \tag{3.6}$$

The boundary conditions eq. (3.5) then become

$$\Phi_s(\mathbf{x}+\ell\hat{x}) = \Phi_s\left(\mathbf{x}+\frac{1}{2}\ell\hat{x}+\frac{\sqrt{3}}{2}\ell\hat{y}\right) = \Phi_s(\mathbf{x})\,. \tag{3.7}$$

Finally, the daughter theory lagrangian becomes

$$\begin{aligned}
L_g &= \frac{1}{16g^2}\sum_{\mathbf{x}} \mathrm{tr}\left[W^\alpha(\mathbf{x})W_\alpha(\mathbf{x})\right]\Big|_{\theta\theta} + \text{h.c.},\\
L_\Phi &= \frac{1}{g^2}\sum_{\mathbf{x}} \mathrm{tr}\left[\overline{\Phi}_1(\mathbf{x})e^{\mathcal{V}(\mathbf{x})}\Phi_1(\mathbf{x})e^{-\mathcal{V}(\mathbf{x}+\mathbf{m}_1\epsilon)} + \overline{\Phi}_2(\mathbf{x})e^{\mathcal{V}(\mathbf{x})}\Phi_2(\mathbf{x})e^{-\mathcal{V}(\mathbf{x}+\mathbf{m}_2\epsilon)} + \right.\\
&\qquad \left. +\overline{\Phi}_3(\mathbf{x})e^{\mathcal{V}(\mathbf{x})}\Phi_3(\mathbf{x})e^{-\mathcal{V}(\mathbf{x}+\mathbf{m}_3\epsilon)}\right]\Big|_{\theta\theta\bar\theta\bar\theta},\\
L_W &= \frac{\sqrt{2}}{g^2}\sum_{\mathbf{x}} \mathrm{tr}\left[\Phi_1(\mathbf{x})\Phi_2(\mathbf{x}+\mathbf{m}_1\epsilon)\Phi_3(\mathbf{x}-\mathbf{m}_3\epsilon) - \right.\\
&\qquad \left. -\Phi_1(\mathbf{x})\Phi_3(\mathbf{x}+\mathbf{m}_1\epsilon)\Phi_2(\mathbf{x}-\mathbf{m}_2\epsilon)\right]\Big|_{\theta\theta} + \text{h.c.},
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
\mathbf{m}_1 &= \hat{x},\\
\mathbf{m}_2 &= -\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\,,\\
\mathbf{m}_3 &= -\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\,.
\end{aligned}$$

Note that the rescaled R-charge vectors satisfy charge neutrality condition:

$$\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3 = 0\,,$$

and the three charge vectors form an equilateral triangle.

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The R-symmetry group of the daughter lattice theory is $G_R^{\text{daughter}} = [\mathrm{U}(1)]^3 \times \mathrm{Spin}(3)$, which is a subgroup of $G_R^{\text{mother}} = \mathrm{Spin}(9)$ surviving after the orbifolding. $[\mathrm{U}(1)]^3$ is individual phase-rotation symmetry of Φ_j ($j = 1, 2, 3$) accompanied by a suitable rotation of θ, compensating the total phase. Spin(3) rotates the three scalars (and their superpartners) in $\mathcal{V}$. The diagonal part of the $[\mathrm{U}(1)]^3$:

$$\Phi_j \to e^{i2\beta/3}\Phi_j \qquad \text{with} \qquad \theta \to e^{-i\beta}\theta \tag{3.9}$$

fits with the interpretation that the emergent space-time after the orbifolding is a two-dimensional lattice — each lattice direction breaks the supersymmetry by one-half, so one-quarter of the sixteen supercharges (transforming under Spin(7) as $\mathbf{8_s}$) would be preserved.

3.3 Continuum and infinite volume limit

We now consider the continuum and infinite volume limits of the theory eq. (3.8). In terms of component fields, eq. (3.8) is expressed as

$$\begin{aligned}
L_g &= \frac{1}{g^2}\sum_{\mathbf{x}} \mathrm{tr}\Big[i\bar\lambda(\mathbf{x})\mathcal{D}_0\lambda(\mathbf{x}) - \bar\lambda(\mathbf{x})\sigma^j[a_j(\mathbf{x}),\lambda(\mathbf{x})] + \frac{1}{2}D(\mathbf{x})^2 + \frac{1}{2}\left(\mathcal{D}_0 a_j(\mathbf{x})\right)^2 + \\
&\qquad\qquad + \frac{1}{4}[a_j(\mathbf{x}), a_l(\mathbf{x})]^2\Big], \\
L_\Phi &= \frac{1}{g^2}\sum_{\mathbf{x}} \mathrm{tr}\Big[\,|F_j(\mathbf{x})|^2 + |\mathcal{D}_0 B_j(\mathbf{x})|^2 + i\overline{\psi}_j(\mathbf{x})\mathcal{D}_0\psi_j(\mathbf{x}) - \\
&\qquad\qquad - \Big|a_l(\mathbf{x})B_j(\mathbf{x}) - B_j(\mathbf{x})a_l(\mathbf{x}+\mathbf{m}_j\epsilon)\Big|^2 - \\
&\quad -\overline{\psi}_j(\mathbf{x})\sigma^l\left\{a_l(\mathbf{x})\psi_j(\mathbf{x}) - \psi_j(\mathbf{x})a_l(\mathbf{x}+\mathbf{m}_j\epsilon)\right\} + \\
&\qquad\qquad + \Big(i\sqrt{2}\overline{B}_j(\mathbf{x})\left\{\lambda(\mathbf{x})\psi_j(\mathbf{x}) - \psi_j(\mathbf{x})\lambda(\mathbf{x}+\mathbf{m}_j\epsilon)\right\} + \text{h.c.}\Big) + \\
&\qquad\qquad + D(\mathbf{x})\left\{B_j(\mathbf{x})\overline{B}_j(\mathbf{x}) - \overline{B}(\mathbf{x}-\mathbf{m}_j\epsilon)B_j(\mathbf{x}-\mathbf{m}_j\epsilon)\right\}\Big], \\
L_W &= \frac{\sqrt{2}}{g^2}\sum_{\mathbf{x}} \mathrm{tr}\Big[F_1(\mathbf{x})\left\{B_2(\mathbf{x}+\mathbf{m}_1\epsilon)B_3(\mathbf{x}-\mathbf{m}_3\epsilon) - B_3(\mathbf{x}+\mathbf{m}_1\epsilon)B_2(\mathbf{x}-\mathbf{m}_2\epsilon)\right\} - \\
&\qquad\qquad - B_1(\mathbf{x})\left\{\psi_2(\mathbf{x}+\mathbf{m}_1\epsilon)\psi_3(\mathbf{x}-\mathbf{m}_3\epsilon) - \psi_3(\mathbf{x}+\mathbf{m}_1\epsilon)\,\psi_2(\mathbf{x}-\mathbf{m}_2\epsilon)\right\} + \\
&\qquad\qquad + (1\to 2,\ 2\to 3,\ 3\to 1) + \\
&\qquad\qquad + (1\to 3,\ 2\to 1,\ 3\to 2)\Big] + \text{h.c.},
\end{aligned} \tag{3.10}$$

where j, l run over 1,2, and 3, a_j are real scalar fields originating from spatial components of the $(3+1)$-dimensional gauge fields before dimensionally reduced to eq. (3.1).

We expand the complex scalar fields B_j around the vacuum configuration $(f/\sqrt{2})\mathbb{1}_k$ as

$$B_j(\mathbf{x}) = \left(\frac{f}{\sqrt{2}}\right)\mathbb{1}_k + B'_j(\mathbf{x}) \qquad \text{with} \qquad f = \sqrt{\frac{2}{3}}\epsilon^{-1}. \tag{3.11}$$

As the expectation value of B_j's is proportional to the unit matrix, the gauge group $\mathrm{U}(k)$ remains unbroken. Thanks to the manifest supersymmetry on the lattice, the degrees of freedom are balanced between bosons and fermions. Thus, it is expected that the theory is free from the problem of fermion doubling. In fact, as pointed out in ref. [13], the vacuum expectation value of B_j's induces fermion bilinear terms (out of the Yukawa coupling terms in L_Φ) which take the form of the Wilson fermion mass term (with a particular value of the Wilson coupling parameter, a point further elaborated in [17]).

Continuum limit is taken with[4]

$$\epsilon \to 0, \qquad \ell \equiv L\epsilon = \text{fixed}, \qquad g_3^2 \equiv \epsilon_x \epsilon_y g^2 = \text{fixed}.$$

The volume of the resulting two-dimensional torus is $\ell \times \frac{\sqrt{3}}{2}\ell$, so, if necessary, an infinite volume limit is attainable by taking $\ell \to \infty$, while the noncommutativity is held fixed [18].

In order to get the theory in the standard form, it is convenient to make the field redefinition

$$B'_j \equiv \frac{1}{\sqrt{3}}\,\mathbf{m}_j \cdot (\mathbf{h} + i\mathbf{v}) + \frac{1}{\sqrt{6}}(h_3 + ih_4)\,, \qquad \overline{B}'_j \equiv \frac{1}{\sqrt{3}}\,\mathbf{m}_j \cdot (\mathbf{h} - i\mathbf{v}) + \frac{1}{\sqrt{6}}\,(h_3 - ih_4)\,,$$
$$\psi_j \equiv \sqrt{\frac{2}{3}}\,\mathbf{m}_j \cdot \mathbf{\Psi} + \frac{1}{\sqrt{3}}\xi\,, \qquad \overline{\psi}_j \equiv \sqrt{\frac{2}{3}}\,\mathbf{m}_j \cdot \overline{\mathbf{\Psi}} + \frac{1}{\sqrt{3}}\xi\,,$$

with

$$\mathbf{h} = \begin{pmatrix} h_x \\ h_y \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}, \qquad \mathbf{\Psi} = \begin{pmatrix} \psi_x \\ \psi_y \end{pmatrix}, \qquad \overline{\mathbf{\Psi}} = \begin{pmatrix} \bar{\psi}_x \\ \bar{\psi}_y \end{pmatrix}.$$

The component fields v_x, v_y, h_x, h_y, h_3, h_4 are hermitian matrices, and ψ_x, ψ_y, ξ are complex 2-component spinors.

Finally, we arrive at the following continuum theory in $(2+1)$-dimensions:

$$L_g = \frac{1}{g_3^2}\int \mathrm{d}^2x\ \mathrm{tr}\left[i\bar{\lambda}\mathcal{D}_0\lambda - \bar{\lambda}\sigma^j[a_j,\lambda] + \frac{1}{2}D^2 + \frac{1}{2}(\mathcal{D}_0 a_j)^2 + \frac{1}{4}[a_j,a_l]^2\right],$$
$$\begin{aligned}L_\Phi = \frac{1}{g_3^2}\int \mathrm{d}^2x\,\mathrm{tr}\Big[&|F_j|^2 + \frac{1}{2}\Big\{(F_{0i})^2 - (\mathcal{D}_i a_j)^2\Big\} + \frac{1}{2}\Big\{(\mathcal{D}_0 h_I)^2 + [a_j,h_I]^2\Big\} + \\ &+ i\overline{\psi}_x\mathcal{D}_0\psi_x + i\overline{\psi}_y\mathcal{D}_0\psi_y + i\overline{\xi}\mathcal{D}_0\xi - i\{\psi_i\mathcal{D}_i\lambda - \overline{\psi}_i\mathcal{D}_i\overline{\lambda}\} - \\ &- \Big\{\overline{\psi}_i\sigma^j[a_j,\psi_i] + i\lambda[h_i,\psi_i] + i\overline{\lambda}[h_i,\overline{\psi}_i]\Big\} - \\ &- \overline{\xi}\sigma^j[a_j,\xi] - \lambda[h_4 + ih_3,\xi] - \overline{\lambda}[h_4 - ih_3,\overline{\xi}] + \\ &+ D\left(\mathcal{D}_x h_x + \mathcal{D}_y h_y - i[h_3,h_4]\right)\Big],\end{aligned}$$
$$\begin{aligned}L_W = \frac{1}{g_3^2}\int d^2x\,\mathrm{tr}\Big[&F_1\frac{1}{\sqrt{6}}\Big\{iF_{xy} + \mathcal{D}_x h_y - \mathcal{D}_y h_x + \sqrt{2}\,\mathcal{D}_y(h_3+ih_4) + [h_x,h_y] + \\ &\qquad + \sqrt{2}\,[h_y,h_3+ih_4]\Big\} + \\ &+ F_2\frac{1}{\sqrt{6}}\Big\{iF_{xy} + \mathcal{D}_x h_y - \mathcal{D}_y h_x - \sqrt{\frac{3}{2}}\,\mathcal{D}_x(h_3+ih_4) -\end{aligned}$$

[4] Then we have to fix two independent radion modes (constant modes of the U(1) part of $\frac{1}{\sqrt{3}}h_x + \frac{1}{\sqrt{6}}h_3$ and h_y) as analyzed by Kaplan et al. [13].

$$
\begin{aligned}
& -\frac{1}{\sqrt{2}}\,\mathcal{D}_y(h_3+ih_4)+[h_x,h_y]- \\
& -\sqrt{\frac{3}{2}}\,[h_x,h_3+ih_4]-\frac{1}{\sqrt{2}}\,[h_y,h_3+ih_4]\Big\}+ \\
+F_3\frac{1}{\sqrt{6}}\Big\{ & iF_{xy}+\mathcal{D}_xh_y-\mathcal{D}_yh_x+\sqrt{\frac{3}{2}}\,\mathcal{D}_x(h_3+ih_4)- \\
& -\frac{1}{\sqrt{2}}\,\mathcal{D}_y(h_3+ih_4)+[h_x,h_y]+ \\
& +\sqrt{\frac{3}{2}}\,[h_x,h_3+ih_4]-\frac{1}{\sqrt{2}}\,[h_y,h_3+ih_4]\Big\}- \\
-\xi\mathcal{D}_x\psi_y+\xi\mathcal{D}_y\psi_x & -\xi[h_x,\psi_y]+\xi[h_y,\psi_x]+\psi_x[h_3+ih_4,\psi_y]\Bigg]+\text{h.c.},
\end{aligned}
\tag{3.12}
$$

where the indices run as $i = x, y$, $I = x, y, 3, 4$. Also, $j, l = 1, 2, 3$, as before. F_{0i}, F_{xy} are the gauge field strength $F_{0i} = \partial_0 v_i - \partial_i v_0 + i[v_0, v_i]$, $F_{xy} = \partial_x v_y - \partial_y v_x + i[v_x, v_y]$, and a_j, h_I stand for 7 scalars. After integrating the auxiliary fields D and F_j's, it is straightforward to see that the theory is the same as the standard form of (2+1) dimensional super-Yang-Mills theory.

In the continuum limit, the supersymmetry is enhanced (quadrupled) and preserves the sixteen supercharges. In eq. (3.12), the manifest R-symmetry is Spin(7), under which the seven scalars a_j, h_I transform as the vector representation **7** and the fermions $\psi_x, \psi_y, \xi, \lambda$ transform as the spinor representation **8**.[5] An important point is that, in the continuum limit, we are referring the R-symmetry Spin(7) to the symmetry involving the *shifted* scalar fields B'_j in eq. (3.11), whereas the R-symmetry $[\mathrm{U}(1)]^3 \times \mathrm{Spin}(3)$ of the lattice daughter theory eq. (3.8) were the one concerning the B_j's descended from the mother theory. Note that the expectation value in eq. (2.6) breaks the diagonal U(1) symmetry eq. (3.9). At a generic vacuum of the continuum theory, where the gauge group is broken to $[\mathrm{U}(1)]^k$, the photon is equivalent (via duality transformation) to a scalar field. Combined with the existing seven scalar fields, the continuum daughter theory would exhibit the continuum limit R-symmetry Spin(8).

4. Super Yang-Mills theory on the noncommutative lattice

We start again with the mother theory eq. (3.1), but now construct a noncommutative version of the lattice super-Yang-Mills theory by a different choice of the orbifold conditions.

4.1 The daughter theory

Here, we take $N = k \cdot mq \cdot nq$, where k, m, n, q are integers, and denote $I := mnq$. We take orbifold conditions for the fields as

$$
\Phi_s = {\omega_{nq}}^{r_{s,a}}\, \Omega_a^\dagger\, \Phi_s\, \Omega_a \qquad \text{for all} \qquad s, a\,, \tag{4.1}
$$

[5]Note that, as Kaplan et al. [13] argued, the four supercharges manifest on the lattice was sufficient to recover the full sixteen supercharges in the continuum limit without any fine-tuning.

where ω_{nq} denotes the (nq)-th root of the unity. We now take the orbifold condition matrices $\Omega_a \in \mathrm{U}(N)$ as[6]

$$\begin{aligned}\Omega_1 &= {U_I}^m \otimes V_q^{\dagger p} \otimes \mathbb{1}_k \\ \Omega_2 &= {V_I}^m \otimes U_q^{\dagger} \otimes \mathbb{1}_k \,.\end{aligned}$$

In the present case, unlike the mutually commutative ones eq. (2.3) that rendered commutative daughter theories, Ω_a's do not commute each other, but obey the 't Hooft-Weyl algebra

$$\Omega_1\Omega_2 = e^{2\pi i\Theta}\Omega_2\Omega_1 \qquad \text{where} \qquad \Theta = \frac{1}{q}\left(p - \frac{m}{n}\right) \quad (\text{mod } 1).$$

The R-charge vectors $\mathbf{r}_s$ in eq. (4.1) are assigned the same as in the table in section 1.

We now introduce $D_a \in \mathrm{U}(N)$ as

$$D_1 = V_I^{\dagger} \otimes \mathbb{1}_q \otimes \mathbb{1}_k \qquad \text{and} \qquad D_2 = U_I \otimes \mathbb{1}_q \otimes \mathbb{1}_k,$$

with the property

$$D_a\Omega_b = {\omega_{nq}}^{-\delta_{ab}}\,\Omega_b D_a.$$

We then define shifted matrix fields $\widetilde{\Phi}_s$ as

$$\Phi_s = \widetilde{\Phi}_s D_1^{r_{s,1}} D_2^{r_{s,2}} \qquad\qquad (s = v, 1, 2, 3)$$

so that $\widetilde{\Phi}_s$'s are subject to *homogeneous* orbifold conditions:

$$\widetilde{\Phi}_s = \Omega_a\widetilde{\Phi}_s\Omega_a^{\dagger} \qquad \text{for all} \qquad s, a\,. \tag{4.2}$$

These orbifold conditions are solvable as follows. Assume that p and q are co-prime, and r and s are integers such that $rp + sq = 1$. We introduce $Z_a \in \mathrm{U}(mq \cdot nq)$ defined as

$$Z_1 = {U_I}^n \otimes V_q^{\dagger} \qquad \text{and} \qquad Z_2 = {V_I}^n \otimes U_q^{\dagger r}$$

with the periodicity:

$$ {Z_1}^L = {Z_2}^L = \mathbb{1}_{mq\cdot nq} \qquad \text{where} \qquad L := mq\,.$$

The Z_a's have algebraic properties that they commute with orbifold condition matrices:

$$[\Omega_a, Z_b] = 0 \qquad \text{for all } a, b\,,$$

but Z_a's do not commute with each other:

$$Z_1Z_2 = e^{-2\pi i\Theta'}Z_2Z_1 \qquad \text{where} \qquad \Theta' = \frac{1}{q}\left(\frac{n}{m} - r\right) \quad (\text{mod } 1)\,.$$

[6]These choices were considered first in [9] for obtaining noncommutative space-time.

We can then construct the complete set of basis for a general solution of the homogeneous orbifold condition eq. (4.2) as

$$J(\mathbf{m}) = e^{-\pi i\Theta' m_1 m_2}\, {Z_2}^{m_2} {Z_1}^{m_1} = J^\dagger(-\mathbf{m})\,. \tag{4.3}$$

Here, m_a's run over $m_a = 0, 1, \ldots, L-1$. We shall be imposing periodic boundary conditions $J(\mathbf{m} + L\hat{a}) = J(\mathbf{m})$ for every a-th direction. Due to the phase-factor $e^{-\pi i\Theta' m_1 m_2}$ introduced in eq. (4.3) for manifest hermiticity, the periodicity requires that $L\Theta'$ is an even-integer. Taking the case L is odd,[7] we can then choose Θ' as

$$\begin{aligned}
\Theta' &= \frac{1}{q}\left(\frac{n}{m} - r\right) \quad (\text{mod } 2) \qquad \text{when } (n - mr) \text{ is even.}\\
\Theta' &= \frac{1}{q}\left(\frac{n}{m} - r\right) + 1 \quad (\text{mod } 2) \qquad \text{when } (n - mr) \text{ is odd.}
\end{aligned}$$

The dual basis to configuration space is obtained via

$$\Delta(\mathbf{n}) = \sum_{m_1, m_2} J(\mathbf{m})\, {\omega_L}^{\mathbf{m}\cdot\mathbf{n}}\,,$$

where again $\mathbf{n} = (n_1, n_2)$ spans a two-dimensional lattice, ranging in each direction over $0, 1, \ldots, L-1$. As is constructed, $\Delta(\mathbf{n})$ is hermitian and periodic $\Delta(\mathbf{n} + L\hat{a}) = \Delta(\mathbf{n})$ along each a-th direction.

To proceed further, we note that the dual basis $\Delta(\mathbf{n})$ satisfies the following identities:

$$\begin{aligned}
\frac{1}{L^2}\sum_{\mathbf{n}} \Delta(\mathbf{n}) &= \mathbb{1}_{mq\cdot nq},\\
D_a\left[\Delta(\mathbf{n}) \otimes \varphi(\mathbf{n})\right] D_a^\dagger &= \Delta(\mathbf{n} - \hat{a}) \otimes \varphi(\mathbf{n}),\\
\widehat{\mathrm{tr}}\left[\Delta(\mathbf{n})\right] &= mq \cdot nq\,,\\
\widehat{\mathrm{tr}}\left[\Delta(\mathbf{n})\Delta(\mathbf{n}')\right] &= \delta_{\mathbf{n},\mathbf{n}'}\, mq \cdot nq\, L^2\,,
\end{aligned} \tag{4.4}$$

where $\varphi(\mathbf{n})$ is arbitrary $k \times k$ matrix. '$\widehat{\mathrm{tr}}$' denotes the trace for $(mq\cdot nq) \times (mq\cdot nq)$ matrices.

As in the commutative case, the dual basis $\Delta(\mathbf{n})$ spans a basis for the subspace complement to the last factor of $(k \times k)$ matrices, so we can decompose the (shifted) matrix variables of the mother theory as

$$\widetilde{\Phi}_s = \frac{1}{L^2}\sum_{\mathbf{n}} \Delta(\mathbf{n}) \otimes \widehat{\Phi}_s(\mathbf{n})\,. \tag{4.5}$$

Here, $\widehat{\Phi}_s$'s are $(k \times k)$ matrix-valued fields defined on the emergent two-dimensional lattice labelled by $\mathbf{n}$. Eq. (4.5) then provides a general solution to the homogeneous orbifold condition eq. (4.2), and hence, after taking into account of shifts through D_a's, to the orbifold condition eq. (4.1).

One last thing we will need to understand concerns how matrix multiplication in the mother theory is translated in the daughter theory. To answer this, consider $N \times N$ matrices

[7] In the case L even, there does not exist Θ' satisfying the periodicity for $(n - mr)$ odd.

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f_1 and f_2 that commute with Ω_a's. As $\Delta(\mathbf{n})$ spans a complete set of basis, the matrices are Fourier-decomposable as in eq. (4.5):

$$f_i = \frac{1}{L^2}\sum_{\mathbf{n}} \Delta(\mathbf{n}) \otimes \widehat{f}_i(\mathbf{n}) \qquad i = 1, 2 .$$

Product of two matrices is again a matrix, so the matrix product $f_1\, f_2$ should be decomposable as above. Utilizing the identities eq. (4.4), we readily find that the product is given by

$$f_1 f_2 = \frac{1}{L^2}\sum_{\mathbf{n}} \Delta(\mathbf{n}) \otimes \left(\widehat{f}_1(\mathbf{n}) \star \widehat{f}_2(\mathbf{n})\right) .$$

In the right-hand-side, product between the two lattice functions $\widehat{f}_1(\mathbf{n})$ and $\widehat{f}_2(\mathbf{n})$ defines (the lattice version of) Moyal's $\star$-product, whose explicit expression is given by

$$\widehat{f}_1(\mathbf{n}) \star \widehat{f}_2(\mathbf{n}) := \left[\widehat{f}_1(\mathbf{n})\widehat{f}_2(\mathbf{n})\right]_\star = \frac{1}{L^2}\sum_{\mathbf{n}',\mathbf{n}''}{}' \, \widehat{f}_1(\mathbf{n}')\widehat{f}_2(\mathbf{n}'')\, e^{2i\widehat{B}(\mathbf{n}-\mathbf{n}')\wedge(\mathbf{n}-\mathbf{n}'')} . \tag{4.6}$$

The parameter $\widehat{B}$ denotes a counterpart of the 'background Neveu-Schwarz B-field' of type-II string theories in the Seiberg-Witten limit [19]:

$$\widehat{B} := \left(\Theta' \frac{L^2}{2\pi}\right)^{-1} . \tag{4.7}$$

The summations over $\mathbf{n}'$ and $\mathbf{n}''$ are restricted to the lattice points satisfying either $\frac{2}{L\Theta'}(n_a - n'_a) \in \mathbb{Z}$ or $\frac{2}{L\Theta'}(n_a - n''_a) \in \mathbb{Z}$.

Rewriting the mother theory action in terms of the hatted configuration space fields, we finally arrive at the following form of the noncommutative daughter theory action:

$$\begin{aligned}
L_g &= \frac{1}{16g^2}\frac{n}{m}\sum_{\mathbf{n}} \mathrm{tr}\left[\widehat{W}^\alpha(\mathbf{n})\widehat{W}_\alpha(\mathbf{n})\right]_\star\Big|_{\theta\theta} + \mathrm{h.c.},\\
L_\Phi &= \frac{1}{g^2}\frac{n}{m}\sum_{\mathbf{n}} \mathrm{tr}\Big[\overline{\widehat{\Phi}}_1(\mathbf{n})e^{\widehat{\mathcal{V}}(\mathbf{n})}\widehat{\Phi}_1(\mathbf{n})e^{-\widehat{\mathcal{V}}(\mathbf{n}+2\widehat{x})} + \overline{\widehat{\Phi}}_2(\mathbf{n})e^{\widehat{\mathcal{V}}(\mathbf{n})}\widehat{\Phi}_2(\mathbf{n})e^{-\widehat{\mathcal{V}}(\mathbf{n}-\widehat{x}+\widehat{y})} + \\
&\qquad + \overline{\widehat{\Phi}}_3(\mathbf{n})e^{\widehat{\mathcal{V}}(\mathbf{n})}\widehat{\Phi}_3(\mathbf{n})e^{-\widehat{\mathcal{V}}(\mathbf{n}-\widehat{x}-\widehat{y})}\Big]_\star\Big|_{\theta\theta\bar{\theta}\bar{\theta}},\\
L_W &= \frac{\sqrt{2}}{g^2}\frac{n}{m}\sum_{\mathbf{n}} \mathrm{tr}\Big[\omega_I^{-1}\widehat{\Phi}_1(\mathbf{n})\widehat{\Phi}_2(\mathbf{n}+2\widehat{x})\widehat{\Phi}_3(\mathbf{n}+\widehat{x}+\widehat{y}) - \\
&\qquad - \omega_I\,\widehat{\Phi}_1(\mathbf{n})\widehat{\Phi}_3(\mathbf{n}+2\widehat{x})\widehat{\Phi}_2(\mathbf{n}+\widehat{x}-\widehat{y})\Big]_\star\Big|_{\theta\theta} + \mathrm{h.c.}.
\end{aligned} \tag{4.8}$$

The daughter theory eq. (4.8) is a counterpart of the daughter theory eq. (3.2). Here, the subscript $\star$'s refer to (lattice version of) Moyal's $\star$-product among all the fields inside the square-bracket. In L_W, due to the noncommutativity between D_1 and D_2, nontrivial phase-factors ω_I^{-1} and ω_I emerged. These phase-factors correspond to a U(1) magnetic flux penetrating through each triangular plaquette defined by each of the two super-potential terms in L_W, and are reminiscent of the discrete torsion in string theory [20] (see also [21]).

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As in the commutative case, we shall take L to be odd, and re-scale the lattice spacings so that (discrete subgroup of) the two-dimensional rotation symmetry is better represented: $\epsilon_x = \frac{1}{2}\epsilon$ and $\epsilon_y = \frac{\sqrt{3}}{2}\epsilon$. Denote coordinates of the resulting equilateral triangular lattice as $\mathbf{x} = (x, y) = (n_x \epsilon_x, n_x \epsilon_y)$, where n_x, n_y range over the values specified in eqs. (3.3), (3.4). Denote also $\ell \equiv L\epsilon$, $\ell_x \equiv L\epsilon_x$, $\ell_y \equiv L\epsilon_y$. The dual basis $\Delta(\mathbf{n})$ is then transcribed in the continuum limit into

$$\Delta(\mathbf{x}) = \sum_{\mathbf{m}} J(\mathbf{m})\, {\omega_{\ell_x}}^{m_1 x} {\omega_{\ell_y}}^{m_2 y} \qquad \text{where} \qquad \omega_{\ell_x} := e^{\frac{2\pi i}{\ell_x}} \quad \omega_{\ell_y} := e^{\frac{2\pi i}{\ell_y}}$$

obeying periodicity:

$$\Delta(\mathbf{x} + \ell\widehat{x}) = \Delta\left(\mathbf{x} + \ell_x \widehat{x} + \ell_y \widehat{y}\right) = \Delta(\mathbf{x})\,.$$

Similarly, the algebraic identities eq. (4.4) are transcribed into:

$$\begin{aligned}
\frac{1}{L^2}\sum_{\mathbf{x}} \Delta(\mathbf{x}) \;&=\; \mathbb{1}_{mq\cdot nq},\\
D_1 \left[\Delta(\mathbf{x}) \otimes \varphi(\mathbf{x})\right] D_1^\dagger \;&=\; \Delta\left(\mathbf{x} - \widehat{x}\epsilon_x\right) \otimes \varphi(\mathbf{x}),\\
D_2 \left[\Delta(\mathbf{x}) \otimes \varphi(\mathbf{x})\right] D_2^\dagger \;&=\; \Delta\left(\mathbf{x} - \widehat{y}\epsilon_y\right) \otimes \varphi(\mathbf{x}),\\
\widehat{\mathrm{tr}}\left[\Delta(\mathbf{x})\right] \;&=\; mq \cdot nq,\\
\widehat{\mathrm{tr}}\left[\Delta(\mathbf{x})\Delta(\mathbf{x}')\right] \;&=\; mq \cdot nq\, L^2\, \delta_{\mathbf{x},\mathbf{x}'}\,,
\end{aligned}$$

for an arbitrary $(k \times k)$ matrix-valued field $\varphi(\mathbf{x})$, and the (shifted) matrix variable decomposition transcribed into fields living on two-dimensional noncommutative space:

$$\widetilde{\Phi}_s = \frac{1}{L^2}\sum_{\mathbf{x}} \Delta(\mathbf{x}) \otimes \Phi_s(\mathbf{x})\,.$$

Re-expressing the daughter theory in terms of these fields, the daughter theory action becomes

$$\begin{aligned}
L_g \;&=\; \frac{1}{16g^2}\frac{n}{m}\sum_{\mathbf{x}} \mathrm{tr}\,\left[W^\alpha(\mathbf{x}) W_\alpha(\mathbf{x})\right]_\star\Big|_{\theta\theta} + \text{h.c.},\\
L_\Phi \;&=\; \frac{1}{g^2}\frac{n}{m}\sum_{\mathbf{x}} \mathrm{tr}\Big[\overline{\Phi}_1(\mathbf{x}) e^{\mathcal{V}(\mathbf{x})} \Phi_1(\mathbf{x}) e^{-\mathcal{V}(\mathbf{x}+\mathbf{m}_1\epsilon)} + \overline{\Phi}_2(\mathbf{x}) e^{\mathcal{V}(\mathbf{x})} \Phi_2(\mathbf{x}) e^{-\mathcal{V}(\mathbf{x}+\mathbf{m}_2\epsilon)} +\\
&\qquad + \overline{\Phi}_3(\mathbf{x}) e^{\mathcal{V}(\mathbf{x})} \Phi_3(\mathbf{x}) e^{-\mathcal{V}(\mathbf{x}+\mathbf{m}_3\epsilon)}\Big]_\star\Big|_{\theta\theta\bar\theta\bar\theta},\\
L_W \;&=\; \frac{\sqrt{2}}{g^2}\frac{n}{m}\sum_{\mathbf{x}} \mathrm{tr}\,\Big[\omega_I^{-1}\, \Phi_1(\mathbf{x}) \Phi_2(\mathbf{x}+\mathbf{m}_1\epsilon) \Phi_3(\mathbf{x}-\mathbf{m}_3\epsilon) -\\
&\qquad - \omega_I\, \Phi_1(\mathbf{x}) \Phi_3(\mathbf{x}+\mathbf{m}_1\epsilon) \Phi_2(\mathbf{x}-\mathbf{m}_2\epsilon)\Big]_\star\Big|_{\theta\theta} + \text{h.c.}
\end{aligned} \tag{4.9}$$

The noncommutative daughter theory eq. (4.9) is a direct counterpart of the commutative daughter theory eq. (3.8). Let us contrast salient features of the latter theory when viewed as a deformation of the former theory. First, product among fields are deformed into

Moyal's $\star$-product. It is readily found that, in terms of continuum variables, Moyal $\star$-product is given by

$$\begin{aligned}\widehat{f}_1(\mathbf{x}) \star \widehat{f}_2(\mathbf{x}) &:= \frac{1}{mq \cdot nq}\, \widehat{\mathrm{tr}}\Big(f_1 f_2 \Delta(\mathbf{x})\Big) \\ &= \frac{1}{L^2} \sum_{\mathbf{x}', \mathbf{x}''}{}' f_1(\mathbf{x}') f_2(\mathbf{x}'')\, e^{2iB(\mathbf{x}-\mathbf{x}')\wedge(\mathbf{x}-\mathbf{x}'')}\,, \end{aligned} \tag{4.10}$$

where

$$B := \left(\Theta' \frac{\ell_x \ell_y}{2\pi}\right)^{-1}$$

and the sums over $\mathbf{x}'$ and $\mathbf{x}''$ are restricted to either

$$\left\{\frac{2}{\ell_x \Theta'}(x - x') \in \mathbb{Z},\, \frac{2}{\ell_y \Theta'}(y - y') \in \mathbb{Z}\right\} \text{ or } \left\{\frac{2}{\ell_x \Theta'}(x - x'') \in \mathbb{Z},\, \frac{2}{\ell_y \Theta'}(y - y'') \in \mathbb{Z}\right\}.$$

We emphasize that the noncommutativity, as defined through Moyal's product, is determined not by Θ but by Θ'. Second, the gauge coupling parameter is re-scaled as

$$g^2 \quad \longrightarrow \quad g^2_{\rm NC} := g^2 \left(\frac{m}{n}\right).$$

Third, as the noncommutative deformation is made, nontrivial phase-factors ω_I^{-1} and ω_I, respectively, are induced on the two terms in the superpotential L_W. One might consider these phase-factors trivial as they seem to disappear in the limit $m, n \to \infty$. In the next subsection, we will find that they actually retain nontrivial effects in the continuum limit.

Note that the nontrivial phase-factors do not affect the R-symmetry of the noncommutative daughter theory *on the lattice.* It is $G_R^{\text{NC daughter}} = [\mathrm{U}(1)]^3 \times \mathrm{Spin}(3)$, same as the one in the commutative counterpart. Again, the diagonal U(1) corresponds to the R-symmetry of the manifest supersymmetry on the two-dimensional noncommutative lattice, as should be the same as that of (2+1)-dimensional $\mathcal{N} = 2$ supersymmetry.

4.2 (Classical) continuum limit

We now take a continuum limit of the lagrangian eq. (4.9):

$$\epsilon \to 0, \quad \ell_{x,y} = L\epsilon_{x,y} = \text{fixed}, \quad m, n \to \infty, \qquad \frac{m}{n} := \mu = \text{fixed}\,. \tag{4.11}$$

The last condition is to ensure the coupling parameter $g^2_{\rm NC}$ and the noncommutativity finite. As we will see shortly, non-trivial phase factors survive in the limit.

We split the superpotential L_W in eq.(4.9) into the two parts $L_W = L_W^{(c)} + L_W^{(s)}$ (not real and imaginary parts), where

$$\begin{aligned} L_W^{(c)} &= \frac{\sqrt{2}}{g^2_{\rm NC}} \left(\cos\frac{2\pi}{I}\right) \sum_{\mathbf{x}} \mathrm{tr}\, \Big[\Phi_1(\mathbf{x})\Phi_2(\mathbf{x} + \mathbf{m}_1\epsilon)\Phi_3(\mathbf{x} - \mathbf{m}_3\epsilon) - \\ &\qquad - \Phi_1(\mathbf{x})\Phi_3(\mathbf{x} + \mathbf{m}_1\epsilon)\Phi_2(\mathbf{x} - \mathbf{m}_2\epsilon)\Big]_\star\Big|_{\theta\theta} + \text{h.c.}\,, \end{aligned} \tag{4.12}$$

$$\begin{aligned} L_W^{(s)} &= \frac{\sqrt{2}}{g^2_{\rm NC}} \left(-i\sin\frac{2\pi}{I}\right) \sum_{\mathbf{x}} \mathrm{tr}\, \Big[\Phi_1(\mathbf{x})\Phi_2(\mathbf{x} + \mathbf{m}_1\epsilon)\Phi_3(\mathbf{x} - \mathbf{m}_3\epsilon) \\ &\qquad + \Phi_1(\mathbf{x})\Phi_3(\mathbf{x} + \mathbf{m}_1\epsilon)\Phi_2(\mathbf{x} - \mathbf{m}_2\epsilon)\Big]_\star\Big|_{\theta\theta} + \text{h.c.} \end{aligned} \tag{4.13}$$

In the continuum limit, structure of L_g, L_Φ, $L_W^{(c)}$ coincide with L_g , L_Φ, L_W in eq. (3.12), except that all the product are replaced by the (continuum version of) Moyal's $\star$-product:

$$f_1(\mathbf{x}) \star f_2(\mathbf{x}) = e^{\frac{i}{2}\widehat{\Theta}'(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1})} f_1(\mathbf{x}) f_2(\mathbf{y})\Big|_{y\to x} \,.$$

A continuum version of the non-commutativity parameter is given by $\widehat{\Theta}'$:

$$\widehat{\Theta}' \equiv \Theta' \frac{\ell_x \ell_y}{2\pi} = \frac{1}{B} \,.$$

We may also take an infinite volume limit while holding the non-commutativity $\widehat{\Theta}'$ finite [18]:

$$\ell_x \ell_y \to \infty, \qquad q \to \infty, \qquad r \ll \mu \,. \tag{4.14}$$

The term $L_W^{(s)}$ proportional to $\sin\frac{2\pi}{I}$, on the other hand, gives rise to a nontrivial term in the continuum limit, and, in the limit eq. (4.14), yields

$$\begin{aligned} L_W^{(s)} &= -i \frac{1}{g_{\mathrm{NC}}^2} \left(\frac{2\pi q}{\ell_x \ell_y} \frac{m}{n} \right) \int \mathrm{d}^2 x \; \frac{1}{\sqrt{6}} \; \mathrm{tr}(F_1 + F_2 + F_3) + \mathrm{h.c.} \\ &:= -i \int \mathrm{d}^2 x \, \mathrm{tr} \Big[\xi_1 \Phi_1 + \xi_2 \Phi_2 + \xi_3 \Phi_3 \Big] \Big|_{\theta\theta} + \mathrm{h.c.} \end{aligned} \tag{4.15}$$

with

$$\xi_1 = \xi_2 = \xi_3 \sim \frac{1}{\widehat{\Theta}'} = B \,.$$

As such, eq. (4.15) represents turning on in the super-potential terms linear in the scalar fields arising from the link variables. Combining with the result for $L_W^{(c)}$, we see that eq. (4.15) represents a magnetic flux background $F_{xy} = -B\mathbb{1}_k$ of the diagonal U(1) gauge group.

One marked difference of the noncommutative daughter theory is that, unlike the commutative counterpart, the lattice supersymmetry is not enhanced to the full sixteen supercharges in the continuum limit, but remains same as that at finite lattice spacing. This may be seen, for example, from symmetry mismatch between eq. (4.12) and eq. (4.15). The cubic superpotential terms in eq. (4.12) are invariant in the continuum limit under enhanced R-symmetry group Spin(4) $\times$ SO(2), which is a subgroup of Spin(6) transforming real and imaginary parts of Φ_j $(j = 1, 2, 3)$ in the defining representation. The individual phase rotations $[\mathrm{U}(1)]^3$ of Φ_j's belong to the R-symmetry group. On the other hand, the linear superpotential terms in eq. (4.15), which descend from eq. (4.13) in the continuum limit, are invariant under the U(1) group, rotating phases of Φ_j's simultaneously:

$$\Phi_j \to e^{i2\beta}\Phi_j \qquad \text{with} \qquad \theta \to e^{-i\beta}\theta \,. \tag{4.16}$$

This rotation (4.16), however, does not belong to (any linear combination of) the $[\mathrm{U}(1)]^3$. As such, the R-symmetry of the continuum theory, consisting of linear and cubic terms in the superpotential, is not present, except Spin(3) keeping Φ_j's intact. It indicates at most four conserved supercharges in the continuum limit.

The definition of Moyal's $\star$-product eq. (4.10) indicates that, in case Θ' is an even-integer, the noncommutative daughter theory reduces to the ordinary commutative one. However, the linear superpotential terms $L_W^{(s)}$ in eq. (4.15) remain non-vanishing, and hence do not have enhancement of the supersymmetry. It just means that the limit $\widehat{\Theta}'$ gives rise to a different commutative theory (with four supercharges only) from the one in eq. (3.12).

4.3 Twist the mother theory!

Consideration of the (classical) continuum limit given above suggests the following possibility. Suppose we start with a 'twisted' version of the mother theory eq. (3.1), where the only difference from eq. (3.1) would be that terms in the superpotential L_W are modified by a phase-factor z:

$$\begin{aligned} L^{\text{twisted}} &= L_g + L_\Phi + L_W^{\text{twisted}}\,, \\ L_W^{\text{twisted}} &= \frac{\sqrt{2}}{g^2}\,\mathrm{Tr}\,\Big[z\Phi_1\Phi_2\Phi_3 - z^{-1}\Phi_1\Phi_3\Phi_2\Big]\Big|_{\theta\theta} + \text{h.c.} \end{aligned}$$

Note that, because of the phase-factors z, z^{-1} introduced, R-symmetry of the 'twisted' mother theory with $z \neq 1$ differs from the untwisted one — the untwisted mothery theory possesses sixteen supercharges, while the twisted mother theory would possess only four. For both theories, after the orbifold conditions eq. (4.1) are imposed, the same four supercharges are retained. Repeating the same analysis as in the previous section, we obtain the following twisted, noncommutative daughter theory:

$$\begin{aligned} L_W^{\text{twisted}} = \frac{\sqrt{2}}{g^2}\frac{n}{m}\sum_{\mathbf{x}} \mathrm{tr}\,\Big[& z\omega_I^{-1}\Phi_1(\mathbf{x})\Phi_2(\mathbf{x}+\mathbf{m}_1\epsilon)\Phi_3(\mathbf{x}-\mathbf{m}_3\epsilon) \\ & -z^{-1}\omega_I\Phi_1(\mathbf{x})\Phi_3(\mathbf{x}+\mathbf{m}_1\epsilon)\Phi_2(\mathbf{x}-\mathbf{m}_2\epsilon)\Big]_\star\Big|_{\theta\theta} + \text{h.c.}\,, \end{aligned}$$

while L_g and L_Φ remains the same as in the untwisted ones in eq. (4.9). For a generic choice of the phase-factor z, the four supercharges manifest on the lattice is not enhanced in the continuum limit. Interestingly, at the special choice of $z = \omega_I$, however, the lattice supersymmetry gets enhanced to preserve the full sixteen supercharges in the continuum limit, as the extra superpotential term eq. (4.15) disappears. The R-symmetry Spin(7) emerges in the continuum limit accordingly, much the same as in the commutative case. Intuitively speaking, the choice $z = \omega_I$ amounts to turning on background gauge field flux along the diagonal U(1) subgroup in the mother theory so that it counter-balances out gauge flux of the daughter theory induced via the linear superpotential terms eq. (4.15).

5. Discussions

In this paper, we have studied gauge theories with manifest supersymmetry on a noncommutative lattice. We have provided a systematic prescription for constructing noncommutative space-time, and have constructed explicitly a $(2+1)$-dimensional noncommutative lattice gauge theory as a daughter theory of $(0+1)$-dimensional mother gauge theory. Extensions to $(d+1)$-dimensional mother theory and to lattices of more than two dimensions are straightforward.

A notable feature of constructing Yang-Mills theories out of matrices via orbifold condition is that rank of the gauge group of the Yang-Mills theories can be taken finite (though the size of the matrices should be necessarily taken to infinity in order to ensure continuum and infinite volume limits). This would be a significant advantage over the traditional Eguchi-Kawai reduction [4], which is limited only to the planar limit of the Yang-Mills theory.

Another novel feature not encountered in the commutative counterpart (section 3) is that the total number of preserved supercharges in the continuum limit depends on a deformation parameter one can introduce to the mother theory. The supercharges in the continuum limit is generically four, the same amount of the supersymmetry as the lattice theory, but is enhanced to sixteen at a particular value of the deformation parameter. As anticipated, the parameter is determined solely by the noncommutativity.

There are several issues deserving further study. One is concerning nonperturbative definition of physical observables. In the continuum formulation and in the super-gravity dual formulation via AdS/CFT correspondence, it was shown [22]–[26] that, due to novelty of the noncommutative gauge invariance, physical observables are 'open Wilson lines', operators local in momentum space but nonlocal in configuration space. It would be very interesting to understand how these open Wilson lines emerge out of the orbifold conditions. Another is regarding potential relation between the noncommutativity and the discrete torsion that has shown up prominently in string theories. Identification of a precise relation would help understanding the discrete torsion intuitively. Finally, results in this work are based on a naive continuum limit dealing with the action itself. To understand the limit in full detail, one would need to understand nontrivial issues of proper renormalization of operators, at least in the lattice perturbation theory [27], in noncommutative gauge theories. Given that our results are rigorous at least in the case that Θ' is even-integer, where the theory reduces to a commutative one (with background gauge flux), one would expect that these issues are amenable with little technical difficulty.

A. Superfield notation

In the Wess-Zumino gauge, we denote the (3+1)-dimensional $\mathcal{N}=1$ superfield (the vector superfield $\mathcal{V}$ and the chiral superfield Φ) in terms of the component fields.

$$\begin{aligned}\mathcal{V}(x) &= -2\theta\sigma^m\bar{\theta}v_m(x) + 2i\theta\theta\bar{\theta}\bar{\lambda}(x) - 2i\bar{\theta}\bar{\theta}\theta\lambda(x) + \theta\theta\bar{\theta}\bar{\theta}D(x),\\ \Phi(x) &= B(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y)\,,\end{aligned}$$

where $y^m = x^m + i\theta\sigma^m\bar{\theta}$, $m = 0,\ldots,3$. v_m are gauge fields and λ, $\bar{\lambda}$ are gaugino. B is a complex scalar field, and ψ is its super-partner. D and F are auxiliary fields.

W_α is a chiral superfield containing the field strength;

$$W_\alpha(x) = -2i\lambda_\alpha(y) + 2\theta_\alpha D(y) - 2i(\sigma^{mn}\theta)_\alpha v_{mn}(y) + 2\theta\theta(\sigma^m\mathcal{D}_m\bar{\lambda}(y))_\alpha\,,$$

where the spinor-index α runs over 1,2. Also

$$\begin{aligned} v_{mn} &= \partial_m v_n - \partial_n v_m + i[v_m, v_n]\,, \\ \mathcal{D}_m \bar{\lambda} &= \partial_m \bar{\lambda} + i[v_m, \bar{\lambda}]\,. \end{aligned}$$

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